FINAL EXAM—SOLUTIONS KEY

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Exercise 1. Recall that

$$\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x] = \operatorname{Spec}\big(\mathbb{Z}[x] \otimes \mathbb{C}\big)$$

Then, $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{A}^1_{\mathbb{C}}$ is the spectrum of:

$$\big(\mathbb{Z}[x] \otimes \mathbb{C}\big) \otimes_{\mathbb{R}} \big(\mathbb{Z}[y] \otimes \mathbb{C}\big) \cong \big(\mathbb{Z}[x] \otimes \mathbb{Z}[y]\big) \otimes (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \cong \mathbb{Z}[x,y] \otimes (\mathbb{C} \times \mathbb{C})$$

where we used that $\mathbb{C} = \mathbb{R}[t]/(t^2+1)$ to see that:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[t]/(t^2+1) = \mathbb{C}[t]/(t-i)(t+i) \cong \mathbb{C} \times \mathbb{C}$$

where the last isomorphism follows from the Chinese reminder theorem.

In this way, we have that $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{A}^1_{\mathbb{C}}$ is isomorphic to the spectrum of

$$\mathbb{Z}[x,y] \otimes (\mathbb{C} \times \mathbb{C}) \cong \big(\mathbb{Z}[x,y] \otimes \mathbb{C}\big) \times \big(\mathbb{Z}[x,y] \otimes \mathbb{C}\big) \cong \mathbb{C}[x,y] \times \mathbb{C}[x,y].$$

In other words, $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{A}^1_{\mathbb{C}}$ is isomorphic to the disjoint union of two copies of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$ and hence disconnected. See [Har77, II, Exercise 2.19].

Exercise 2. Since k is algebraically closed, the points of $\mathbb{A}^1 = \operatorname{Spec} k[t]$ correspond to the ideals $(t-a) \subset k[t]$ with $a \in k$; which are the closed points, and the zero ideal $0 \subset k[t]$; which defines the generic point η. Let's denote the point of $\mathbb{A}^1$ corresponding to $(t-a)$ by x_a.

The fiber of f at x_a is the spectrum of

$$k[x,y,z] \otimes_{k[t]} k[t]/(t-a) \cong k[x,y,z]/(x^2+y-a) \cong k[x,z]$$

which is an integral domain. Therefore, the fiber of f at x_a is irreducible and so has exactly one irreducible component.

It remains to analyze the generic fiber. The fiber of f at η is the spectrum of the following ring

$$k[x,y,z] \otimes_{k[t]} k(t) \cong W^{-1}k[x,y,z]$$

where $W \subset k[x,y,z]$ is the multiplicative system given by nonzero polynomials in x^2+y with coefficients in k.[1] Since localizations of integral domains are integral domains,[2] the generic fiber is irreducible and so has only one irreducible component.

Summing up, f has no non-irreducible fiber. In other words, all fibers of f have exactly one irreducible component.

Exercise 3.

[1] That is, W is the image of $k[t] \smallsetminus 0$ under λ. It's important to notice that λ is injective.

[2] In fact, $W^{-1}k[x,y,z]$ is a nonzero subring of $k(x,y,z)$

(1). Suppose, for the sake of contradiction, that M_p is not torsion-free. Then, there are $0 \neq a/f \in A_p$ and $0 \neq m/g \in M_p$ such that

$$\frac{a}{f} \cdot \frac{m}{g} = \frac{am}{fg} = 0 \in M_p,$$

which implies that there is $h \in A \smallsetminus p$ such that

$$ham = 0 \in M.$$

Since M is torsion-free and $m \neq 0$, we conclude that $ha = 0$. However, this implies that $a/f = 0 \in A_p$; which is a contradiction.

(2). Since the problem is local, we may assume that $X = \operatorname{Spec} A$ where A is a noetherian local normal domain of dimension 1 (this uses part (1)). That is, we may assume that A is a DVR and so a PID.[3] Further, we may say $\mathscr{F} = \tilde{M}$ that where M is a finitely generated A-module that is torsion-free. By using the structure theorem for finitely generated modules over PIDs, we conclude that M is free; as required.

(3). Let $X = \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$ and $\mathscr{F} = \tilde{M}$ where $M = (x, y) \subset \mathbb{C}[x, y]$. Note that M is clearly torsion-free (as $\mathbb{C}[x, y]$ is an integral domain). However, M is not locally free. Indeed, since X is connected, if M were locally free the function $\mathfrak{p} \mapsto \varphi(\mathfrak{p}) = \dim_{k(\mathfrak{p})} M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ would be constant on $\operatorname{Spec} \mathbb{C}[x, y]$; see [Har77, II, Exercise 5.8. (c)]. However, $\varphi\big((0)\big) = 1$ whereas $\varphi\big((x, y)\big) = 2$.

Exercise 4.

(1). Consider the following exact sequence of graded S-modules:

$$0 \longrightarrow S(-d) \xrightarrow{\cdot f} S \longrightarrow S/f \longrightarrow 0$$

Applying the exact functor $\widetilde{(-)}$ to it yields the required exact sequence:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n_k}(-d) \longrightarrow \mathscr{O}_{\mathbb{P}^n_k} \longrightarrow \iota_* \mathscr{O}_X \longrightarrow 0 \tag{0.1}$$

Recall that, by construction [Har77, II, Exercise 3.12, Corollary 5.16 (a) and its proof], for all homogeneous ideal $I \subset S$, applying $\widetilde{(-)}$ to the graded quotient $S \longrightarrow S/I$ yields $j^{\#}$ where j is the induced closed immersion $j \colon \operatorname{Proj} S/I \longrightarrow \operatorname{Proj} S$.[4] More generally, one has $j_*(\tilde{N}) \cong \widetilde{{}_S N}$; see [Har77, II, Proof of Proposition 5.12 (c)].

(2). Recall that (by definition) $\iota^* \mathscr{O}_{\mathbb{P}^n_k}(1) = \mathscr{O}_X(1)$ and so $\iota^* \mathscr{O}_{\mathbb{P}^n_k}(r) = \mathscr{O}_X(r)$ for all r. Hence, twisting (0.1) by $\mathscr{O}_{\mathbb{P}^n_k}(r)$ and using the projection formula yields the following exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n_k}(r-d) \longrightarrow \mathscr{O}_{\mathbb{P}^n_k}(r) \longrightarrow \iota_* \mathscr{O}_X(r) \longrightarrow 0 \tag{0.2}$$

Indeed, one uses the projection formula [Har77, II, Exercise 5.1. (d)] to obtain the following isomorphism:[5]

$$\iota_* \mathscr{O}_X \otimes \mathscr{O}_{\mathbb{P}^n_k}(r) \cong \iota_*\big(\mathscr{O}_X \otimes \iota^* \mathscr{O}_{\mathbb{P}^n_k}(r)\big) = \iota_*\big(\mathscr{O}_X \otimes \mathscr{O}_X(r)\big) = \iota_* \mathscr{O}_X(r).$$

[3] The normality hypothesis is very important here. In fact, the statement doesn't hold without it. For instance, consider $A = k[x, y]/x^2 - y^3$ and $M = (x, y)$.

[4] Indeed, $\tilde{I}$ is the ideal sheaf corresponding to the closed immersion determined by j, which means that $\mathscr{O}_{\operatorname{Proj} S}/\tilde{I} = \widetilde{S/I}$ is $j_* \mathscr{O}_{\operatorname{Proj} S/I}$.

[5] One could also use [Har77, II, Proposition 5.12 (c)].

The short exact sequence (0.2) yields the following long exact sequence on cohomologies

$$\cdots \longrightarrow H^i\big(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(r-d)\big) \longrightarrow H^i\big(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(r)\big) \longrightarrow H^i\big(\mathbb{P}^n_k, \iota_*\mathscr{O}_X(r)\big) \longrightarrow H^{i+1}\big(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(r-d)\big) \longrightarrow \cdots$$

Now, using the simultaneous vanishings

$$H^i\big(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(r)\big) = 0 \quad \text{and} \quad H^{i+1}\big(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(r-d)\big) = 0$$

for all $0 < i < n-1$, we conclude that

$$H^i\big(\mathbb{P}^n_k, \iota_*\mathscr{O}_X(r)\big) = 0$$

for all $0 < i < n-1$. However,

$$H^i\big(\mathbb{P}^n_k, \iota_*\mathscr{O}_X(r)\big) = H^i(X, \mathscr{O}_X(r)),$$

say by [Har77, III, Lemma 2.10]. This solves the exercise.

References

[Har77] R. Hartshorne: *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)